Relative Fixed Divisors of the Type-A Casimir on N-ality Congruence Sectors

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Abstract

Let $\mathfrak{g} = A_{n-1}$, let $d \mid n$, and restrict the integralized shifted quadratic Casimir to dominant weights whose box number is fixed modulo d. We study the gcd of all pairwise Casimir differences in this arithmetic congruence sector. For $n \geq 5$, we prove the stable formula

$$\Delta_{n,d,c} = d \gcd\Big(n + d + 2c,\, 2\gcd\Big(d, \frac{n}{d}\Big)\Big),$$

and show that the stable relative divisor is attained already at affine level 3, uniformly in n, d, c. This degree bound is sharp: $\kappa_{5,5,1} = 3$. The proof combines a quadratic lattice-coset realization lemma, an exact type-A Casimir formula, equal-box certificates, and cross-box congruence certificates. We also record the resulting order of the subgroup generated by relative Kac–Peterson phases on the same arithmetic sector. The result is formulated independently of any identification with the full primary spectrum of a non-simply connected WZW model.

Keywords. Quadratic Casimir; fixed divisors; lattice cosets; N-ality; affine Lie algebras; WZW modular data.

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1 Introduction

Fixed divisors of integer-valued polynomials form a classical subject in arithmetic algebra. General frameworks for fixed divisors on arbitrary subsets of Dedekind domains, and multivariate extensions, are available in the work of Bhargava and subsequent authors; see [1, 9, 8]. The present paper studies a particular quadratic polynomial on a distinguished family of non-Cartesian congruence cosets arising from type-A representation theory.

Let

$$\mathfrak{g} = A_{n-1}, \qquad n \geq 5,$$

and normalize the invariant form by requiring long roots to have squared length 2. If

$$\lambda = \sum_{i=1}^{n-1} s_i \omega_i \qquad (s_i \in \mathbb{N} := \{0, 1, 2, \dots\}),$$

then the affine integrability condition at level k is

$$\sum_{i=1}^{n-1} s_i \leq k.$$

The associated Young diagram has box number

$$\mathrm{box}(\lambda) = \sum_{i=1}^{n-1} i\, s_i.$$

Modulo n, this is the usual N-ality of a type-A highest weight. Here we retain only its residue modulo an arbitrary divisor $d \mid n$.

Let

$$Q_2(\lambda) = (\lambda, \lambda) + 2(\rho, \lambda)$$

be the shifted quadratic Casimir. Since the least positive integer which makes Q_2 integral on the type-A weight lattice is n, we set

$$F_n(\lambda) = nQ_2(\lambda).$$

For a fixed congruence class $c \pmod d$, we consider the gcd of all differences

$$F_n(\lambda) - F_n(\mu)$$

among integrable dominant weights with box number congruent to $c \pmod d$. The main result gives both the stable gcd and a uniform finite level at which it is already attained.

The theorem is arithmetic. Its final application to Kac–Peterson phases is only an application of the usual modular-data formula restricted to an explicitly defined subset of weights. In particular, we do not identify this subset with the full spectrum of a WZW theory on $SU(n)/\mathbb{Z}_d$; such an identification requires further simple-current and global-form input [4, 5].

2 Definitions and main theorem

For

$$s = (s_1, \ldots, s_{n-1}) \in \mathbb{N}^{n-1}, \qquad \lambda(s) = \sum_{i=1}^{n-1} s_i \omega_i,$$

write

$$\mathrm{box}(s) = \sum_{i=1}^{n-1} i\, s_i.$$

Let $d \mid n$ and $0 \le c < d$.

Definition 2.1. The arithmetic N-ality congruence sector at affine level k is

$$\mathcal{S}_{n,d,c}(k) = \left\{ s \in \mathbb{N}^{n-1} : \sum_{i=1}^{n-1} s_i \le k, \quad \mathrm{box}(s) \equiv c \pmod d \right\}.$$

Its relative fixed divisor is

$$D^{\mathrm{rel}}_{n,d,c}(k) = \gcd \left\{ F_n(s) - F_n(t) : s, t \in \mathcal{S}_{n,d,c}(k) \right\}.$$

The stable relative divisor is

$$\Delta_{n,d,c} = \gcd \left\{ F_n(s) - F_n(t) : \begin{array}{c} s, t \in \mathbb{N}^{n-1}, \\ \mathrm{box}(s) \equiv \mathrm{box}(t) \equiv c \pmod d \end{array} \right\}.$$

Finally, define the stabilization threshold

$$\kappa_{n,d,c} = \min \left\{ k \ge 1 : D^{\mathrm{rel}}_{n,d,c}(m) = \Delta_{n,d,c} \text{ for every } m \ge k \right\}.$$

Theorem 2.2 (Uniform degree-three stabilization)**.** *For every*

$$n \geq 5, \qquad d \mid n, \qquad 0 \leq c < d,$$

one has

$$\boxed{D^{\mathrm{rel}}_{n,d,c}(3) = \Delta_{n,d,c} = d \gcd\Big(n + d + 2c,\ 2 \gcd\Big(d, \frac{n}{d}\Big)\Big).}$$

Consequently,

$$\boxed{\kappa_{n,d,c} \leq 3.}$$

Corollary 2.3 (Local form)**.** *For every prime p,*

$$\nu_p(\Delta_{n,d,c}) = \nu_p(d) + \min\Big\{\nu_p(n + d + 2c),\ \nu_p(2) + \min\Big(\nu_p(d), \nu_p\Big(\frac{n}{d}\Big)\Big)\Big\}.$$

Corollary 2.4 (Sharpness)**.** *The bound in Theorem 2.2 is sharp:*

$$\boxed{\kappa_{5,5,1} = 3.}$$

3 The integralized Casimir in Young-diagram coordinates

Associate with $s \in \mathbb{N}^{n-1}$ the partition Λ whose columns have height i with multiplicity s_i. Write its row lengths as

$$\Lambda_1 \geq \cdots \geq \Lambda_n = 0,$$

and set

$$B = |\Lambda| = \sum_{i=1}^{n-1} i\, s_i.$$

Define

$$\kappa(\Lambda) = \sum_{a=1}^{n} \Lambda_a(\Lambda_a + 1 - 2a).$$

Lemma 3.1 (Integralized Casimir formula)**.** *One has*

$$\boxed{F_n(\Lambda) = n^2 B + n\kappa(\Lambda) - B^2.}$$

Moreover,

$$\kappa(\Lambda) \in 2\mathbb{Z},$$

and therefore

$$\boxed{F_n(\Lambda) \equiv n^2 B - B^2 \pmod{2n}.}$$

Proof. In the standard Euclidean realization of the A_{n-1} weight lattice,

$$Q_2(\Lambda) = \sum_{a=1}^{n} \Lambda_a(\Lambda_a + n + 1 - 2a) - \frac{B^2}{n}.$$

Since

$$\sum_{a=1}^{n} \Lambda_a(\Lambda_a + n + 1 - 2a) = nB + \kappa(\Lambda),$$

multiplication by n proves the formula. Finally,

$$\Lambda_a(\Lambda_a + 1) \in 2\mathbb{Z} \qquad \text{and} \qquad 2a\Lambda_a \in 2\mathbb{Z},$$

so $\kappa(\Lambda)$ is even. $\square$

4 Quadratic polynomials on lattice cosets

Let
$$f : \mathbb{Z}^r \longrightarrow \mathbb{Z}$$
be an integer-valued polynomial of degree at most 2, and define its polarization by
$$\mathcal{B}_f(x, y) = f(x + y) - f(x) - f(y) + f(0).$$
Let $L \subseteq \mathbb{Z}^r$ be a full-rank sublattice and $\xi \in \mathbb{Z}^r$. Put
$$\mathscr{I}_{\xi,L}(f) = \langle f(x) - f(y) : x, y \in \xi + L\rangle_{\mathbb{Z}}$$
and
$$\mathscr{I}^{+}_{\xi,L}(f) = \langle f(x) - f(y) : x, y \in (\xi + L) \cap \mathbb{N}^r\rangle_{\mathbb{Z}}\,.$$
Lemma 4.1 (Positive realization of a quadratic coset ideal)**.** *One has*
$$\boxed{\mathscr{I}_{\xi,L}(f) = \mathscr{I}^{+}_{\xi,L}(f).}$$

Proof. Let $b_1, \dots, b_r$ be a $\mathbb{Z}$-basis of L, and write
$$A_i = f(\xi + b_i) - f(\xi), \qquad C_{ii} = \mathcal{B}_f(b_i, b_i), \qquad C_{ij} = \mathcal{B}_f(b_i, b_j)\ (i < j).$$
For
$$g(t) = f\left(\xi + \sum_{i=1}^{r} t_i b_i\right) - f(\xi),$$
the quadratic Newton expansion is
$$g(t) = \sum_{i=1}^{r} A_i t_i + \sum_{i=1}^{r} C_{ii}\binom{t_i}{2} + \sum_{i<j} C_{ij} t_i t_j.$$
It follows that
$$\mathscr{I}_{\xi,L}(f) = \langle A_i, C_{ii}, C_{ij} : 1 \le i \le r,\ 1 \le i < j \le r\rangle_{\mathbb{Z}}\,. \tag{4.1.1}$$
Choose $m \ge 1$ with $m\mathbb{Z}^r \subseteq L$, take M sufficiently large, and set
$$q = Mm(1, \dots, 1) \in L, \qquad z = \xi + q.$$
All of
$$z, \quad z + b_i, \quad z + 2b_i, \quad z + b_i + b_j$$
lie in $\mathbb{N}^r$. Therefore
$$C_{ij} = f(z + b_i + b_j) - f(z + b_i) - f(z + b_j) + f(z)$$
and
$$C_{ii} = f(z + 2b_i) - 2f(z + b_i) + f(z)$$
belong to $\mathscr{I}^{+}_{\xi,L}(f)$.

Writing $q = \sum_j t_j b_j$, we also have
$$f(z + b_i) - f(z) = A_i + \mathcal{B}_f(q, b_i),$$
where $\mathcal{B}_f(q, b_i)$ is an integer combination of the already obtained polar generators. Thus every A_i belongs to $\mathscr{I}^{+}_{\xi,L}(f)$. By (4.1.1), this proves
$$\mathscr{I}_{\xi,L}(f) \subseteq \mathscr{I}^{+}_{\xi,L}(f),$$
and the reverse inclusion is immediate. $\square$

5 The stable divisor

Define the charge lattice

$$L_d = \left\{ x \in \mathbb{Z}^{n-1} : \sum_{i=1}^{n-1} i\, x_i \equiv 0 \pmod{d} \right\}$$

and put $\xi_c = ce_1$. Then $\xi_c + L_d$ is the full Dynkin-coordinate coset with box number congruent to $c \pmod{d}$. A $\mathbb{Z}$-basis of L_d is

$$b_1 = de_1, \qquad b_i = e_i - ie_1 \quad (2 \le i \le n-1).$$

Let $\mathcal{B}_n$ denote the polarization of F_n.

Lemma 5.1 (Polar form)**.** *For $1 \le i, j \le n-1$,*

$$\boxed{\mathcal{B}_n(e_i, e_j) = 2\min(i,j)(n - \max(i,j)).}$$

Proof. For type A_{n-1},

$$n(\omega_i, \omega_j) = \min(i,j)(n - \max(i,j)).$$

The quadratic part of $F_n = nQ_2$ is $n(\lambda, \lambda)$, whose polarization is twice the associated bilinear form. □

Lemma 5.2 (Exact polar gcd)**.** *For $n \ge 4$,*

$$\boxed{\gcd\{\mathcal{B}_n(b_i, b_j) : 1 \le i \le j \le n-1\} = 2\gcd(n, d^2).}$$

Proof. A direct calculation from Lemma 5.1 gives

$$\mathcal{B}_n(b_1, b_1) = 2d^2(n-1),$$

$$\mathcal{B}_n(b_1, b_i) = -2dn(i-1) \qquad (2 \le i \le n-1),$$

and

$$\mathcal{B}_n(b_i, b_j) = 2n\max(i,j)(\min(i,j) - 1) \qquad (i, j \ge 2).$$

Every polar value is divisible by $2\gcd(n, d^2)$. Conversely, for $n \ge 4$, the values

$$2d^2(n-1), \qquad 4n, \qquad 6n$$

occur. Hence

$$\begin{aligned} \gcd(2d^2(n-1), 4n, 6n) &= \gcd(2d^2(n-1), 2n) \\ &= 2\gcd(d^2(n-1), n) \\ &= 2\gcd(n, d^2), \end{aligned}$$

because $\gcd(n, n-1) = 1$. □

Proposition 5.3 (Stable divisor formula)**.** *For $n \ge 4$,*

$$\boxed{\Delta_{n,d,c} = d\gcd\left(n + d + 2c,\ 2\gcd\left(d, \frac{n}{d}\right)\right).}$$

Proof. By Lemma 4.1, the stable difference ideal is generated by the increments along the above basis and by the polar values. The increments are

$$F_n(\xi_c + b_1) - F_n(\xi_c) = (n-1)d(n+d+2c)$$

and

$$F_n(\xi_c + b_i) - F_n(\xi_c) = -2nc(i-1) \qquad (2 \le i \le n-1).$$

The latter increments have gcd $2nc$. Lemma 5.2 therefore yields

$$\Delta_{n,d,c} = \gcd\big((n-1)d(n+d+2c),\ 2nc,\ 2\gcd(n,d^2)\big). \tag{5.3.1}$$

Write $n = dm$ and $g = \gcd(d,m)$. Then

$$\gcd(n,d^2) = dg, \qquad 2dg \mid 2nc,$$

so (5.3.1) becomes

$$\Delta_{n,d,c} = d\gcd\big((n-1)(n+d+2c), 2g\big).$$

For every odd prime factor of g, one has $p \nmid n-1$, since $g \mid n$. If n is even, then $n-1$ is odd. If n is odd, then d is odd and $n+d+2c$ is even. Thus multiplication by $n-1$ does not alter the gcd with $2g$, and the stated formula follows. □

Proof of Corollary 2.3. Take p-adic valuations in Proposition 5.3. □

6 Level-three certificates

We now construct level-three pairs whose Casimir differences generate the stable divisor.

6.1 Equal-box certificates

For the indicated admissible values of a, define

$$\begin{gathered}
\mathsf{P}_4(a) = (\omega_a + \omega_2,\ \omega_a + 2\omega_1), \qquad 2 \le a \le n-1, \\
\mathsf{P}_6(a) = (\omega_a + \omega_3,\ \omega_a + \omega_2 + \omega_1), \qquad 3 \le a \le n-1, \\
\mathsf{S}_4 = (\omega_2, 2\omega_1), \qquad \mathsf{S}_6 = (\omega_3, \omega_2 + \omega_1), \\
\mathsf{Q}_4(a) = (\omega_a + \omega_3 + \omega_1,\ \omega_a + 2\omega_2), \qquad 3 \le a \le n-1, \\
\mathsf{Q}_6(a) = (\omega_a + \omega_4 + \omega_1,\ \omega_a + \omega_3 + \omega_2), \qquad 4 \le a \le n-1,
\end{gathered}$$

and

$$\mathsf{R}_4 = (\omega_3 + \omega_1, 2\omega_2), \qquad \mathsf{R}_6 = (\omega_4 + \omega_1, \omega_3 + \omega_2).$$

Lemma 6.1 (Explicit equal-box identities)**.** *The pairs above have the following common box numbers and Casimir differences:*

pair	*partition pair*	*common box number*	$\Delta\kappa$
$\mathsf{P}_4(a)$	$(2,2,1^{a-2}),\ (3,1^{a-1})$	$a+2$	-4
$\mathsf{P}_6(a)$	$(2,2,2,1^{a-3}),\ (3,2,1^{a-2})$	$a+3$	-6
S_4	$(1,1),\ (2)$	2	-4
S_6	$(1,1,1),\ (2,1)$	3	-6
$\mathsf{Q}_4(a)$	$(3,2,2,1^{a-3}),\ (3,3,1^{a-2})$	$a+4$	-4
$\mathsf{Q}_6(a)$	$(3,2,2,2,1^{a-4}),\ (3,3,2,1^{a-3})$	$a+5$	-6
R_4	$(2,1,1),\ (2,2)$	4	-4
R_6	$(2,1,1,1),\ (2,2,1)$	5	-6

Hence the corresponding Casimir differences are, respectively, $-4n$ and $-6n$.

Proof. The row-wise values of $\Delta\kappa$ follow by direct substitution into the definition of κ. Because the two partitions in every row have the same box number, Lemma 3.1 gives

$$F_n(\Lambda) - F_n(\Gamma) = n(\kappa(\Lambda) - \kappa(\Gamma)),$$

which proves the claim. □

Lemma 6.2 (Uniform equal-box bound)**.** *For every $n \geq 5$, $d \mid n$, and $0 \leq c < d$, the sector $\mathcal{S}_{n,d,c}(3)$ contains a pair with Casimir difference $-4n$ and a pair with Casimir difference $-6n$. Consequently,*

$$\boxed{D^{\mathrm{rel}}_{n,d,c}(3) \mid 2n.}$$

Proof. First assume $d < n$. Then $d \leq n/2$. The intervals

$$[4, n+1] \qquad \text{and} \qquad [6, n+2]$$

contain at least d consecutive integers. Choose

$$B_4 \in [4, n+1], \qquad B_6 \in [6, n+2]$$

with $B_4 \equiv B_6 \equiv c \pmod{d}$. Put

$$a_4 = B_4 - 2, \qquad a_6 = B_6 - 3.$$

Then $\mathsf{P}_4(a_4)$ and $\mathsf{P}_6(a_6)$ lie in the required sector and supply $-4n$ and $-6n$.

Now assume $d = n$. For $6 \leq c \leq n-1$, use $\mathsf{P}_4(c-2)$ and $\mathsf{P}_6(c-3)$. For the remaining residues, use

c	$-4n$	$-6n$
0	$\mathsf{P}_4(n-2)$	$\mathsf{P}_6(n-3)$
1	$\mathsf{P}_4(n-1)$	$\mathsf{P}_6(n-2)$
2	S_4	$\mathsf{P}_6(n-1)$
3	$\mathsf{Q}_4(n-1)$	S_6
4	R_4	$\mathsf{Q}_6(n-1)$
5	$\mathsf{P}_4(3)$	R_6.

For $(n, c) = (5, 0)$, replace $\mathsf{P}_6(n-3)$ by R_6. Lemma 6.1 verifies both the box residues and the displayed differences. Therefore the relative divisor divides both $4n$ and $6n$, hence divides $2n$. □

6.2 Cross-box certificates

For $0 \leq B \leq 3(n-1)$, write

$$B = q(n-1) + r, \qquad 0 \leq r < n-1,$$

and define

$$\Pi(B) = q\omega_{n-1} + \begin{cases} \omega_r, & r > 0, \\ 0, & r = 0. \end{cases}$$

Then $\Pi(B)$ has box number B and affine level at most 3.

Lemma 6.3 (Cross-box gcd certificate)**.** *Let $g = \gcd(d, n/d)$. If*

$$c + 2d \leq 3(n-1),$$

and

$$A_t = F_n(\Pi(c+td)) - F_n(\Pi(c)) \qquad (t = 1, 2),$$

then

$$\boxed{\gcd(2n, A_1, A_2) = d\gcd(n + d + 2c, 2g).}$$

Proof. Put $\phi_n(B) = n^2B - B^2$. Lemma 3.1 gives

$$F_n(\Pi(B)) \equiv \phi_n(B) \pmod{2n}.$$

Hence

$$A_t \equiv X_t \pmod{2n}, \qquad X_t = td(n^2 - 2c - td).$$

Writing $n = dm$, we obtain

$$\begin{aligned} \gcd(2n, X_1, X_2) &= d\gcd(2m, n^2 - 2c - d, 2(n^2 - 2c - 2d)) \\ &= d\gcd(2m, 2d, n^2 - 2c - d) \\ &= d\gcd(2g, n^2 - 2c - d). \end{aligned} \tag{6.3.1}$$

Write $d = gd_0$ and $m = gm_0$. Then $n = g^2d_0m_0$, so

$$2g \mid n(n+1).$$

Indeed,

$$\frac{n(n+1)}{2g} = \frac{gd_0m_0(n+1)}{2} \in \mathbb{Z},$$

because either gd_0m_0 is even or else n is odd and $n+1$ is even. Since

$$(n^2 - 2c - d) + (n + d + 2c) = n(n+1),$$

we have

$$\gcd(2g, n^2 - 2c - d) = \gcd(2g, n + d + 2c).$$

Substitution into (6.3.1) completes the proof. □

Lemma 6.4 (Terminal cases)**.** *If*

$$c + 2d > 3(n-1),$$

then $d = n$*,* $c \in \{n-2, n-1\}$*, and*

$$\Delta_{n,n,c} = 2n.$$

Proof. If $d < n$, then $d \le n/2$ and

$$c + 2d \le 3d - 1 \le \frac{3n}{2} - 1 \le 3(n-1),$$

a contradiction. Thus $d = n$. The strict inequality implies $c > n-3$, hence $c \in \{n-2, n-1\}$. Proposition 5.3 then gives

$$\Delta_{n,n,c} = n\gcd(2n + 2c, 2) = 2n.$$

□

7 Proof of the main theorem

Proof of Theorem 2.2. By definition,

$$\Delta_{n,d,c} \mid D^{\mathrm{rel}}_{n,d,c}(3). \tag{2.2.1}$$

By Lemma 6.2,

$$D^{\mathrm{rel}}_{n,d,c}(3) \mid 2n. \tag{2.2.2}$$

Suppose first that $c + 2d \le 3(n-1)$. The two cross-box differences A_1, A_2 occur in the level-three sector. Hence, by Lemma 6.3,

$$D^{\mathrm{rel}}_{n,d,c}(3) \mid \gcd(2n, A_1, A_2) = d \gcd\Big(n + d + 2c, 2\gcd\Big(d, \frac{n}{d}\Big)\Big) = \Delta_{n,d,c}.$$

Together with (2.2.1), this proves equality.

Suppose instead that $c + 2d > 3(n-1)$. Lemma 6.4 gives $\Delta_{n,d,c} = 2n$. Combining (2.2.1) and (2.2.2) yields

$$2n = \Delta_{n,d,c} \mid D^{\mathrm{rel}}_{n,d,c}(3) \mid 2n,$$

so equality again follows.

For $m \ge 3$, the inclusion

$$\mathcal{S}_{n,d,c}(3) \subseteq \mathcal{S}_{n,d,c}(m)$$

implies

$$D^{\mathrm{rel}}_{n,d,c}(m) \mid D^{\mathrm{rel}}_{n,d,c}(3).$$

The stable divisor divides every finite-level divisor. Hence

$$\Delta_{n,d,c} \mid D^{\mathrm{rel}}_{n,d,c}(m) \mid \Delta_{n,d,c},$$

which proves $\kappa_{n,d,c} \le 3$. □

Proof of Corollary 2.4. At level 2,

$$\mathcal{S}_{5,5,1}(2) = \{\omega_1,\ \omega_2 + \omega_4,\ 2\omega_3\}.$$

Their integralized Casimir values are

$$F_5(\omega_1) = 24, \qquad F_5(\omega_2 + \omega_4) = 64, \qquad F_5(2\omega_3) = 84.$$

Therefore

$$D^{\mathrm{rel}}_{5,5,1}(2) = \gcd(40, 60) = 20.$$

On the other hand,

$$\Delta_{5,5,1} = 5\gcd(12, 2) = 10.$$

Theorem 2.2 gives $D^{\mathrm{rel}}_{5,5,1}(3) = 10$, and hence $\kappa_{5,5,1} = 3$. □

8 Relative Kac–Peterson phases

For $\lambda, \mu \in P^+_k$, the standard Kac–Peterson diagonal T-matrix gives the relative phase formula

$$\frac{T_{\lambda\lambda}}{T_{\mu\mu}} = \exp\Big(2\pi i \frac{F_n(\lambda) - F_n(\mu)}{2n(k+n)}\Big);$$

see [7]. Restricting this formula to the arithmetic sector yields the following.

Corollary 8.1 (Relative phase order)**.** *For $n \ge 5$, $d \mid n$, $0 \le c < d$, and $k \ge 3$, the subgroup generated by*

$$\Big\{\frac{T_{\lambda\lambda}}{T_{\mu\mu}} : \lambda, \mu \in \mathcal{S}_{n,d,c}(k)\Big\}$$

has order

$$\boxed{\frac{2n(k+n)}{\gcd(2n(k+n),\, d\gcd(n + d + 2c, 2\gcd(d, \frac{n}{d})))}}.$$

Proof. Let $N = 2n(k+n)$. The order of the subgroup generated by $\exp(2\pi i a/N)$, as a ranges over the relevant Casimir differences, is

$$\frac{N}{\gcd\left(N, D^{\mathrm{rel}}_{n,d,c}(k)\right)}.$$

Theorem 2.2 identifies the divisor for every $k \geq 3$. $\square$

9 Scope and bibliographic positioning

Remark 9.1 (Scope of the WZW interpretation). The sector

$$\mathcal{S}_{n,d,c}(k) = \{\lambda \in P_k^+ : \mathrm{box}(\lambda) \equiv c \pmod{d}\}$$

is an explicitly defined arithmetic subset of integrable weights. Theorem 2.2 concerns this set, and Corollary 8.1 concerns relative Kac–Peterson phases restricted to it. No assertion is made that this set is automatically the complete primary spectrum of a WZW model with target group $SU(n)/\mathbb{Z}_d$. Such a statement requires extra conditions involving the admissible level, simple-current selection rules, orbit identification, fixed-point resolution, and potentially discrete torsion; see, for example, [4, 5].

Remark 9.2 (Bibliographic positioning). The general fixed-divisor literature cited in the introduction supplies broad frameworks for polynomial values on subsets and in several variables. The result proved here concerns a specific type-A Casimir polynomial and a specific non-Cartesian congruence kernel. This draft makes no claim of bibliographic priority for the formula or the degree-three stabilization bound.

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