Affine Casimir Fixed Divisors, p-Local Stabilization, Weighted Matching Formulas, and WZW Relative Twists

Abstract

For a complex simple Lie algebra $\mathfrak{g}$, we study the fixed divisor of the integralized shifted quadratic Casimir polynomial on the integrable affine alcove P_k^+. We prove an exact weighted-Newton formula expressing this divisor at every level k through the active Newton coefficients of the polynomial. This yields a p-local description of the minimal stabilization level. We then give a complete classification for all simple types, derive lattice and discriminant interpretations of the denominator and stable content, and express the p-local coefficients through paths and weighted matching polynomials of Dynkin diagrams. The resulting formulas require no explicit inverse Cartan matrix. Finally, under the standard Kac–Peterson phase formula, the fixed-divisor calculation gives the projective order of the WZW T-matrix. The manuscript states the results without a claim of bibliographic priority; a separate literature audit is required before any novelty claim.

Keywords. Simple Lie algebras; affine Weyl alcoves; integer-valued polynomials; fixed divisors; Cartan matrices; matching polynomials; WZW modular data; p-adic valuation.
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1 Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra and let P_k^+ be the set of dominant integral weights integrable at affine level k. The shifted quadratic Casimir

$$Q_2(\lambda) = (\lambda, \lambda) + 2(\rho, \lambda)$$

controls the relative WZW twist phase. After multiplying by the smallest denominator $\ell_{\mathfrak{g}}$ making Q_2 integral on the weight lattice, we obtain an integer-valued quadratic polynomial

$$F_{\mathfrak{g}} = \ell_{\mathfrak{g}} Q_2.$$

The object studied here is the fixed divisor

$$D_{\mathfrak{g}}(k) = \gcd\{F_{\mathfrak{g}}(\lambda) : \lambda \in P_k^+\}.$$

The first purpose of the paper is to replace finite enumeration of P_k^+ by an exact formula in terms of Newton coefficients. The second is to classify the resulting stabilization thresholds and their p-local refinements. The third is to express the relevant coefficients directly through paths and matching polynomials of the Dynkin tree. The final WZW application uses the classical modular formula of Kac and Peterson [5]; background on root data and affine Lie algebras is standard [1, 4].

The language of integer-valued polynomials and binomial bases is classical [2]. Likewise, matching expansions of determinants on trees are standard combinatorial phenomena. The point here is the combination of these ingredients for the shifted Casimir on affine alcoves, together with exact p-local stabilization certificates. We do not assert that the statements below are new in the literature.

2 Root data, affine alcoves, and fixed divisors

Let $\mathfrak{g}$ be simple of rank r. We normalize the invariant form by

$$(\theta_{\mathrm{long}}, \theta_{\mathrm{long}}) = 2.$$

Let P be the weight lattice, P^+ the dominant cone, $\omega_1, \dots, \omega_r$ the fundamental weights, and ρ the half-sum of the positive roots. Since $\langle \rho, \alpha_i^\vee \rangle = 1$, one has

$$\rho = \omega_1 + \cdots + \omega_r.$$

Write the highest coroot as

$$\theta^\vee = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee,$$

where $a_i^\vee$ are the comarks. For $k \geq 1$,

$$P_k^+ = \left\{ \lambda = \sum_{i=1}^{r} s_i \omega_i \in P^+ : \sum_{i=1}^{r} a_i^\vee s_i \leq k \right\}.$$

Definition 2.1. Define

$$\ell_{\mathfrak{g}} = \min\{\ell \in \mathbb{Z}_{>0} : \ell Q_2(P) \subseteq \mathbb{Z}\}, \qquad F_{\mathfrak{g}} = \ell_{\mathfrak{g}} Q_2,$$

$$D_{\mathfrak{g}}(k) = \gcd\{F_{\mathfrak{g}}(\lambda) : \lambda \in P_k^+\}, \qquad \gcd\{0\} = 0,$$

and

$$\delta_{\mathfrak{g}} = \gcd\{F_{\mathfrak{g}}(\lambda) : \lambda \in P^+\}.$$

The minimal stabilization level is

$$k_0(\mathfrak{g}) = \min\{k \geq 1 : D_{\mathfrak{g}}(m) = \delta_{\mathfrak{g}} \text{ for every } m \geq k\}.$$

For

$$\lambda(s) = \sum_{i=1}^{r} s_i \omega_i,$$

put

$$u_i = F_{\mathfrak{g}}(\omega_i), \qquad v_i = F_{\mathfrak{g}}(2\omega_i) - 2F_{\mathfrak{g}}(\omega_i),$$

$$w_{ij} = F_{\mathfrak{g}}(\omega_i + \omega_j) - F_{\mathfrak{g}}(\omega_i) - F_{\mathfrak{g}}(\omega_j) \qquad (i < j).$$

Then

$$v_i = 2\ell_{\mathfrak{g}}(\omega_i, \omega_i), \qquad w_{ij} = 2\ell_{\mathfrak{g}}(\omega_i, \omega_j).$$

3 Weighted Newton coefficients on discrete simplices

For $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}_{>0}^r$, set

$$S_k(\mathbf{a}) = \{s \in \mathbb{N}^r : \mathbf{a} \cdot s \leq k\}.$$

For a multi-index $\alpha \in \mathbb{N}^r$, write

$$\binom{s}{\alpha} = \prod_{i=1}^{r} \binom{s_i}{\alpha_i}.$$

Theorem 3.1 (Weighted Newton fixed-divisor theorem)**.** *Let*

$$f(s) = \sum_{\alpha \in \mathbb{N}^r} c_\alpha \binom{s}{\alpha} \in \mathrm{Int}(\mathbb{Z}^r, \mathbb{Z})$$

be its finite multivariate binomial expansion. Then, for every $k \geq 0$,

$$\boxed{\langle f(s) : s \in S_k(\mathbf{a}) \rangle_{\mathbb{Z}} = \langle c_\alpha : \mathbf{a} \cdot \alpha \leq k \rangle_{\mathbb{Z}} \,.}$$

Proof. Let I_k be the ideal generated by the values of f on $S_k(\mathbf{a})$ and let J_k be the ideal on the right. If $s \in S_k(\mathbf{a})$ and $\binom{s}{\alpha} \neq 0$, then $\alpha_i \leq s_i$ for every i, hence $\mathbf{a} \cdot \alpha \leq \mathbf{a} \cdot s \leq k$. Thus $f(s) \in J_k$ and $I_k \subseteq J_k$.

Conversely,

$$c_\alpha = (\Delta^\alpha f)(0) = \sum_{\beta \leq \alpha} (-1)^{|\alpha| - |\beta|} \binom{\alpha}{\beta} f(\beta).$$

If $\mathbf{a} \cdot \alpha \leq k$, then $\mathbf{a} \cdot \beta \leq k$ for every $\beta \leq \alpha$. Hence $f(\beta) \in I_k$, so $c_\alpha \in I_k$. Therefore $J_k \subseteq I_k$. □

Corollary 3.2 (Exact affine-Casimir formula)**.** *For every $k \geq 1$,*

$$\boxed{D_{\mathfrak{g}}(k) = \gcd(\{u_i : a_i^\vee \leq k\}, \{v_i : 2a_i^\vee \leq k\}, \{w_{ij} : a_i^\vee + a_j^\vee \leq k\}).}$$

In particular,

$$\boxed{\delta_{\mathfrak{g}} = \gcd(\{u_i\}_i, \{v_i\}_i, \{w_{ij}\}_{i<j}).}$$

Proof. The quadratic polynomial $F_{\mathfrak{g}}(\lambda(s))$ has the binomial expansion

$$F_{\mathfrak{g}}(\lambda(s)) = \sum_i u_i s_i + \sum_i v_i \binom{s_i}{2} + \sum_{i<j} w_{ij} s_i s_j.$$

Its only nonzero binomial coefficients are u_i, v_i, and w_{ij}, of respective affine weights $a_i^\vee$, $2a_i^\vee$, and $a_i^\vee + a_j^\vee$. Apply Theorem 3.1. □

Corollary 3.3 (p-local stabilization)**.** *Let p be prime and put $\nu_p(0) = +\infty$. Then*

$$\nu_p(D_{\mathfrak{g}}(k)) = \min(\{\nu_p(u_i) : a_i^\vee \leq k\}, \{\nu_p(v_i) : 2a_i^\vee \leq k\}, \{\nu_p(w_{ij}) : a_i^\vee + a_j^\vee \leq k\}).$$

If $e_{\mathfrak{g},p} = \nu_p(\delta_{\mathfrak{g}})$, define

$$k_{0,p}(\mathfrak{g}) = \min \begin{cases} a_i^\vee, & \nu_p(u_i) = e_{\mathfrak{g},p}, \\ 2a_i^\vee, & \nu_p(v_i) = e_{\mathfrak{g},p}, \\ a_i^\vee + a_j^\vee, & \nu_p(w_{ij}) = e_{\mathfrak{g},p}. \end{cases}$$

Then

$$\boxed{k_0(\mathfrak{g}) = \max_p k_{0,p}(\mathfrak{g}).}$$

Proof. The first statement is the p-adic valuation of the gcd in Corollary 3.2. The stable valuation is the global minimum $e_{\mathfrak{g},p}$; it is first attained when a coefficient with this valuation becomes active. Simultaneous stabilization at all primes gives the final formula. □

Proposition 3.4 (Denominator criterion)**.** *Let $L \in \mathbb{Z}_{>0}$. The polynomial LQ_2 is integral on P if and only if*

$$LQ_2(\omega_i), \qquad 2L(\omega_i, \omega_i), \qquad 2L(\omega_i, \omega_j) \quad (i < j)$$

are all integers. Consequently, $\ell_{\mathfrak{g}}$ is the least common multiple of the reduced denominators of these rational numbers.

Proof. Apply Theorem 3.1 formally to the quadratic polynomial LQ_2 on all of $\mathbb{Z}^r$. Its binomial coefficients are precisely the three displayed families. Integral coefficients are equivalent to integral values on the full lattice, and conversely the coefficients are obtained from values by finite differences. □

Remark 3.5. A correct certificate of minimality is the strict inequality

$$\nu_p(D_{\mathfrak{g}}(k)) > \nu_p(\delta_{\mathfrak{g}}),$$

not merely the existence of a prime dividing $D_{\mathfrak{g}}(k)$ but not $\delta_{\mathfrak{g}}$.

4 Discriminant and lattice interpretations

Theorem 4.1 (Affine discriminant denominator)**.** *Let*

$$q_{\rho,\mathfrak{g}}(\lambda) = Q_2(\lambda) + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$$

and define

$$\Lambda_{\rho,\mathfrak{g}} = \{\mu \in P : 2(\mu, P) \subseteq \mathbb{Z},\ Q_2(\mu) \in \mathbb{Z}\}.$$

Then $q_{\rho,\mathfrak{g}}$ descends to a function

$$\overline{q}_{\rho,\mathfrak{g}} : P/\Lambda_{\rho,\mathfrak{g}} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

and

$$\boxed{\ell_{\mathfrak{g}} = \mathrm{ord}(\overline{q}_{\rho,\mathfrak{g}}).}$$

Proof. For $\mu \in \Lambda_{\rho,\mathfrak{g}}$,

$$Q_2(\lambda + \mu) - Q_2(\lambda) = 2(\lambda, \mu) + Q_2(\mu) \in \mathbb{Z},$$

so the function descends. By definition, $\ell_{\mathfrak{g}}$ is the smallest positive integer killing every value of Q_2 modulo $\mathbb{Z}$, which is precisely the order of the induced finite-valued function. □

Let

$$W = ((\omega_i, \omega_j))_{1 \le i,j \le r}, \qquad H_{\mathfrak{g}} = \ell_{\mathfrak{g}} W, \qquad B_{\mathfrak{g}} = 2H_{\mathfrak{g}}.$$

Theorem 4.2 (Parity criterion)**.** *The following are equivalent:*

$$2 \mid \delta_{\mathfrak{g}},$$

$$H_{\mathfrak{g}} \in M_r(\mathbb{Z}) \quad \textit{and} \quad (H_{\mathfrak{g}})_{ii} \in 2\mathbb{Z} \textit{ for all } i,$$

$$(P, \ell_{\mathfrak{g}}(\cdot,\cdot)) \quad \textit{is an integral even lattice.}$$

Proof. One has

$$v_i = 2(H_{\mathfrak{g}})_{ii}, \qquad w_{ij} = 2(H_{\mathfrak{g}})_{ij},$$

$$u_i = (H_{\mathfrak{g}})_{ii} + 2\sum_j (H_{\mathfrak{g}})_{ij}.$$

Thus divisibility by 2 of every v_i and w_{ij} is equivalent to integrality of $H_{\mathfrak{g}}$, and then $u_i \equiv (H_{\mathfrak{g}})_{ii} \pmod 2$. This is exactly the evenness criterion for the rescaled weight lattice. □

Theorem 4.3 (Newton–Smith formula at odd primes)**.** *Let*

$$\sigma_1(B_{\mathfrak{g}}) = \gcd\{(B_{\mathfrak{g}})_{ij} : 1 \le i, j \le r\},$$

the first invariant factor of the Smith normal form of $B_{\mathfrak{g}}$. Then

$$\boxed{\delta_{\mathfrak{g}} = \gcd(\sigma_1(B_{\mathfrak{g}}), u_1, \dots, u_r).}$$

For every odd prime p,

$$\boxed{\nu_p(\delta_{\mathfrak{g}}) = \nu_p(\sigma_1(B_{\mathfrak{g}})).}$$

Proof. The entries of $B_{\mathfrak{g}}$ are the coefficients v_i and w_{ij}, hence the first formula follows from Corollary 3.2. If $p \neq 2$, then in $\mathbb{Z}_{(p)}$ the scalar 2 is invertible and

$$u_i = \frac{(B_{\mathfrak{g}})_{ii}}{2} + \sum_j (B_{\mathfrak{g}})_{ij}$$

belongs to the ideal generated by all entries of $B_{\mathfrak{g}}$. Therefore the linear coefficients cannot lower the p-adic content at odd p. □

5 Classification of the affine-Casimir fixed divisor

Theorem 5.1 (Complete classification)**.** *For every complex simple Lie algebra, the values of $\ell_{\mathfrak{g}}$, $\delta_{\mathfrak{g}}$, $k_0(\mathfrak{g})$, and the exceptional low-level values of $D_{\mathfrak{g}}(k)$ are given in Table 1. Here $g_r = \gcd(r, 4)$.*

Table 1: Fixed divisors and minimal stabilization levels.

Type	$h^\vee$	$\ell_{\mathfrak{g}}$	$\delta_{\mathfrak{g}}$	$k_0(\mathfrak{g})$	Low-level values
A_r	$r+1$	$r+1$	$\gcd(r,2)$	2	$D(1) = (r+2)\gcd(r,2)$
B_r	$2r-1$	$4/g_r$	1	1 if $r = 2, 4$; else 2	$D(1) = r/g_r$
C_r	$r+1$	2	1	1	$D(k) = 1$ for all k
D_r	$2r-2$	$4/g_r$	1	2	$D(1) = 2r-1$
E_6	12	3	2	3	$D(1) = 52,\ D(2) = 4$
E_7	18	2	1	2	$D(1) = 57$
E_8	30	1	2	5	$D(1) = 0,\ D(2) = D(3) = 12,\ D(4) = 4$
F_4	9	1	1	3	$D(1) = 12,\ D(2) = 2$
G_2	4	3	2	3	$D(1) = 12,\ D(2) = 4$

Proof. The proof is organized by coefficient certificates. Proposition 3.4 proves the listed denominators from the explicit coefficient formulas recorded below and in Appendix A. Corollary 3.2 shows that it is enough to identify the active Newton coefficients, while Corollary 3.3 proves minimality through the displayed p-adic witnesses. The complete certificates are recorded in Tables 2 and 3.

For A_r, with $h = r+1$,

$$u_i = (r+2)i(h-i), \qquad v_i = 2i(h-i), \qquad w_{ij} = 2i(h-j) \quad (i<j).$$

All comarks are 1. Thus $D_{A_r}(1) = (r+2)\gcd_i i(h-i) = (r+2)\gcd(r,2)$. At level 2 every Newton coefficient is active. If r is even, all coefficients are even and $w_{1r} = 2$; if r is odd, $u_1 = r(r+2)$ is odd while $w_{1r} = 2$. This gives the stated value and $k_0(A_r) = 2$.

For B_r, let $d = \gcd(r, 4)$ and $L = 4/d$. With comarks $(1, 2, \ldots, 2, 1)$,

$$u_1 = \frac{8r}{d}, \qquad u_r = \frac{r(2r+1)}{d}, \qquad w_{1r} = \frac{4}{d}.$$

At level 1 the active values are u_1, u_r, whose gcd is r/d. At level 2, w_{1r} is active and $\gcd(u_r, w_{1r}) = 1$. This proves the B_r line.

For C_r, all comarks are 1 and

$$u_1 = 2r + 1, \qquad u_2 = 4r.$$

They are already active at level 1 and are coprime. Hence $D_{C_r}(k) = 1$ for all k.

For D_r, with $d = \gcd(r, 4)$,

$$u_1 = \frac{4(2r-1)}{d}, \qquad u_r = \frac{r(2r-1)}{d}, \qquad w_{1r} = \frac{4}{d}.$$

The level-one gcd is $2r - 1$, while the level-two coefficient w_{1r} makes the gcd equal to 1.

The exceptional rows follow from the exact matrices and active-coefficient lists in Appendix A. The p-adic witnesses in Table 3 prove that no smaller stabilization level is possible. $\square$

Table 2: Classical coefficient certificates. The stated pairs have gcd equal to the stable content.

Type	Level one	Stabilizing certificate	Minimality witness
A_r	$(r+2)\gcd(r,2)$	$w_{1r} = 2$ and $u_1 = r(r+2)$	$D(1) \neq \delta$
B_r	$r/\gcd(r,4)$	$\gcd(u_r, w_{1r}) = 1$	any $p \mid r/\gcd(r,4)$
C_r	$\gcd(2r+1, 4r) = 1$	already level 1	none
D_r	$2r - 1$	$\gcd(u_r, w_{1r}) = 1$	any $p \mid (2r-1)$

Table 3: Exceptional active-coefficient certificates.

Type	$\delta_{\mathfrak{g}}$	Low-level gcds	Stabilizing certificate	Minimality witness
E_6	2	$52, 4$	$\gcd(u_1, w_{12}) = \gcd(52, 10) = 2$	$\nu_2(D(2)) = 2 > 1$
E_7	1	57	$\gcd(u_5, u_6) = \gcd(112, 57) = 1$	$\nu_3(D(1)) = 1$
E_8	2	$0, 12, 12, 4$	$\gcd(u_8, w_{18}) = \gcd(144, 10) = 2$	$\nu_3(D(3)) = 1,\ \nu_2(D(4)) = 2$
F_4	1	$12, 2$	$\gcd(v_1, w_{12}) = \gcd(2, 3) = 1$	$\nu_2(D(2)) = 1$
G_2	2	$12, 4$	$\gcd(v_1, w_{12}) = \gcd(4, 6) = 2$	$\nu_2(D(2)) = 2 > 1$

6 Paths, cofactor formulas, and weighted matchings

Let $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ and $d_i = (\alpha_i, \alpha_i)/2$. Put $R = \operatorname{diag}(d_1, \ldots, d_r)$ and $\Delta_{\mathfrak{g}} = \det A$. The underlying Dynkin diagram Γ is a tree.

For an induced forest $F \subseteq \Gamma$, define

$$\Phi_F = \sum_{M \in \mathcal{M}(F)} (-1)^{|M|} 2^{|V(F)| - 2|M|} \prod_{\{u,v\} \in M} a_{uv} a_{vu}.$$

For vertices i, j, let $P_{ij} = (i = i_0, \ldots, i_s = j)$ be the unique path, put

$$\Phi_{ij} = \Phi_{\Gamma \setminus V(P_{ij})}, \qquad \pi_{j \to i} = \prod_{t=0}^{s-1} (-a_{i_{t+1}, i_t}),$$

and use $\pi_{i \to i} = 1$, $\Phi_{ii} = \Phi_{\Gamma \setminus \{i\}}$.

Theorem 6.1 (Cartan cofactors from weighted matching polynomials)**.** *For every induced forest $F \subseteq \Gamma$,*

$$\boxed{\Phi_F = \det A_F.}$$

Moreover,

$$\boxed{\operatorname{adj}(A)_{ji} = \pi_{j \to i} \Phi_{ij}.}$$

If $B_{\mathfrak{g}} = 2\ell_{\mathfrak{g}} R A^{-T}$, then

$$\boxed{v_i = \frac{2\ell_{\mathfrak{g}} d_i}{\Delta_{\mathfrak{g}}} \Phi_{ii}, \qquad w_{ij} = \frac{2\ell_{\mathfrak{g}} d_i}{\Delta_{\mathfrak{g}}} \pi_{j \to i} \Phi_{ij}.}$$

Finally, with

$$\Psi_i = \Phi_{ii} + 2 \sum_{j=1}^{r} \pi_{j \to i} \Phi_{ij},$$

one has

$$\boxed{u_i = \frac{\ell_{\mathfrak{g}} d_i}{\Delta_{\mathfrak{g}}} \Psi_i.}$$

Proof. In the Leibniz expansion of $\det A_F$, a nonzero permutation can have only fixed points and transpositions: a longer cycle would give a cycle in the forest. A fixed point contributes 2 and a transposition (u, v) contributes $-a_{uv} a_{vu}$. Thus the expansion is exactly Φ_F.

For the cofactor, delete row i and column j. Every nonzero term must use the unique path P_{ij}; its off-diagonal entries give $\pi_{j \to i}$, while the remaining vertices give the matching determinant Φ_{ij}. Since

$$B_{\mathfrak{g}} = 2\ell_{\mathfrak{g}} R A^{-T} = \frac{2\ell_{\mathfrak{g}} R \operatorname{adj}(A)^T}{\Delta_{\mathfrak{g}}},$$

the formulas for v_i and w_{ij} follow. The formula for u_i follows from

$$u_i = \frac{v_i}{2} + \sum_j (B_{\mathfrak{g}})_{ij}.$$

□

Theorem 6.2 (p-local weighted-path formula)**.** *Let $e_{\mathfrak{g},p} = \nu_p(\delta_{\mathfrak{g}})$. Define*

$$\lambda_{i,p}^{\mathrm{lin}} = \nu_p(\ell_{\mathfrak{g}} d_i) + \nu_p(\Psi_i) - \nu_p(\Delta_{\mathfrak{g}}),$$

$$\lambda_{i,p}^{\mathrm{diag}} = \nu_p(2\ell_{\mathfrak{g}} d_i) + \nu_p(\Phi_{ii}) - \nu_p(\Delta_{\mathfrak{g}}),$$

$$\lambda_{ij,p}^{\mathrm{mix}} = \nu_p(2\ell_{\mathfrak{g}} d_i) + \nu_p(\pi_{j \to i}) + \nu_p(\Phi_{ij}) - \nu_p(\Delta_{\mathfrak{g}}).$$

Then

$$\nu_p(u_i) = \lambda_{i,p}^{\mathrm{lin}}, \quad \nu_p(v_i) = \lambda_{i,p}^{\mathrm{diag}}, \quad \nu_p(w_{ij}) = \lambda_{ij,p}^{\mathrm{mix}},$$

and

$$\boxed{k_{0,p}(\mathfrak{g}) = \min \begin{cases} a_i^\vee, & \lambda_{i,p}^{\mathrm{lin}} = e_{\mathfrak{g},p}, \\ 2a_i^\vee, & \lambda_{i,p}^{\mathrm{diag}} = e_{\mathfrak{g},p}, \\ a_i^\vee + a_j^\vee, & \lambda_{ij,p}^{\mathrm{mix}} = e_{\mathfrak{g},p}. \end{cases}}$$

Proof. Take p-adic valuations in Theorem 6.1, then apply Corollary 3.3. □

7 Parity matching, odd-prime rigidity, and the E_8 extremum

Theorem 7.1 (Parity depth from perfect matchings)**.** *For $\mathfrak{g} = A_{2m}, E_6, E_8$, the Dynkin tree has a unique perfect matching M, and*

$$\boxed{k_{0,2}(\mathfrak{g}) = \min\{a_i^\vee + a_j^\vee : i < j,\ \Gamma \setminus V(P_{ij}) \text{ has a perfect matching}\}.}$$

Equivalently, the minimum is taken over the paths P_{ij} alternating with respect to M.

Proof. In the three stated types, $\delta_{\mathfrak{g}} = 2$ and $\ell_{\mathfrak{g}}$ is odd. Directly from the matrices in Appendix A (and from the A_{2m} formula), one has $u_i/2 \equiv 0 \pmod 2$ and $v_i/2 \equiv 0 \pmod 2$. Hence the first coefficient achieving stable valuation 1 must be a mixed coefficient w_{ij}.

Modulo 2, the matching determinant of a forest retains only perfect-matchings terms. A forest has at most one perfect matching. Therefore $w_{ij}/2$ is odd exactly when the residual forest $\Gamma \setminus V(P_{ij})$ has a perfect matching. For a tree with unique perfect matching this is equivalent to alternation of the removed path. □

Corollary 7.2 (Diagrammatic explanation of the E_8 delay)**.**

$$k_{0,2}(A_{2m}) = 2, \qquad k_{0,2}(E_6) = 3, \qquad \boxed{k_{0,2}(E_8) = 5.}$$

Proof. For E_8 in the numbering

$$1 - 2 - 3 - 4 - 5 - 6 - 7, \qquad 8 \text{ attached to } 3,$$

a perfect matching is

$$\{\{1,2\},\{3,8\},\{4,5\},\{6,7\}\}.$$

The least comark weight of an alternating path is $2 + 3 = 5$, attained for example by the paths with endpoints $(1,8)$, $(6,7)$, and $(7,8)$. □

Theorem 7.3 (Odd-prime rigidity)**.** *For every odd prime p,*

$$k_{0,p}(\mathfrak{g}) \le 2,$$

except for

$$\boxed{k_{0,3}(E_8) = 4.}$$

More precisely,

$$k_{0,p}(A_r) = \begin{cases} 2, & p \mid (r+2), \\ 1, & p \nmid (r+2), \end{cases} \qquad k_{0,p}(B_r) = \begin{cases} 2, & p \mid r, \\ 1, & p \nmid r, \end{cases}$$

$$k_{0,p}(C_r) = 1, \qquad k_{0,p}(D_r) = \begin{cases} 2, & p \mid (2r-1), \\ 1, & p \nmid (2r-1), \end{cases}$$

$$k_{0,p}(E_6) = \begin{cases} 2, & p = 13, \\ 1, & p \ne 13, \end{cases} \qquad k_{0,p}(E_7) = \begin{cases} 2, & p \in \{3, 19\}, \\ 1, & p \notin \{3, 19\}, \end{cases}$$

$$k_{0,p}(E_8) = \begin{cases} 4, & p = 3, \\ 2, & p \ne 3, \end{cases} \quad k_{0,p}(F_4) = \begin{cases} 2, & p = 3, \\ 1, & p \ne 3, \end{cases} \quad k_{0,p}(G_2) = \begin{cases} 2, & p = 3, \\ 1, & p \ne 3. \end{cases}$$

Proof. The classification table and Corollary 3.3 suffice. For the classical series, the only odd primes surviving at level one are those dividing $r+2$, r, or $2r-1$ as displayed, and the level-two certificates in Table 2 remove them. The exceptional rows are read from Table 3. The sole delay beyond level two occurs for E_8 at $p=3$: the active linear coefficients up to level three are $96, 60, 120, 144$, all divisible by 3, while $u_2 = 196$ becomes active at level four and is nonzero modulo 3. □

Theorem 7.4 (Unique E_8 extremum)**.** *For every complex simple Lie algebra,*

$$\boxed{k_0(\mathfrak{g}) \leq 5.}$$

Moreover,

$$\boxed{k_0(\mathfrak{g}) = 5 \iff \mathfrak{g} = E_8.}$$

Proof. The classical series satisfy $k_0 \leq 2$. The exceptional values are $k_0(E_6) = 3$, $k_0(E_7) = 2$, $k_0(E_8) = 5$, $k_0(F_4) = 3$, and $k_0(G_2) = 3$, by Theorem 5.1. The equality for E_8 follows from Corollary 7.2 and Theorem 7.3. □

8 WZW relative twists

We now assume the standard Kac–Peterson relative-phase formula [5]

$$\frac{T_{\lambda\lambda}}{T_{00}} = \exp\left(2\pi i \frac{Q_2(\lambda)}{2(k+h^\vee)}\right) \qquad (\lambda \in P_k^+).$$

Theorem 8.1 (Projective order of the relative WZW twist)**.** *For every simple* $\mathfrak{g}$ *and every* $k \geq 1$*,*

$$\boxed{\mathrm{ord}_{\mathrm{proj}}(T_{\mathfrak{g},k}) = \frac{2\ell_{\mathfrak{g}}(k+h^\vee)}{\gcd(2\ell_{\mathfrak{g}}(k+h^\vee), D_{\mathfrak{g}}(k))}.}$$

Proof. Put $N = 2\ell_{\mathfrak{g}}(k+h^\vee)$. Then

$$\frac{T_{\lambda\lambda}}{T_{00}} = \exp\left(2\pi i \frac{F_{\mathfrak{g}}(\lambda)}{N}\right).$$

The projective order is the least $n > 0$ with $N \mid nF_{\mathfrak{g}}(\lambda)$ for every $\lambda \in P_k^+$. Since the ideal generated by these values is $D_{\mathfrak{g}}(k)\mathbb{Z}$, Bezout's identity reduces the condition to $N \mid nD_{\mathfrak{g}}(k)$. The least such n is $N/\gcd(N, D_{\mathfrak{g}}(k))$. □

Remark 8.2. The order of the categorical twist and the Frobenius–Schur exponent are standard modular-category invariants [6]. The fixed divisor $D_{\mathfrak{g}}(k)$ is finer than the resulting relative twist order: the latter sees only $\gcd(2\ell_{\mathfrak{g}}(k+h^\vee), D_{\mathfrak{g}}(k))$.

A Exceptional Cartan data and exact certificates

For E_6, E_7, E_8 we use the diagrams

$$E_6 : 1-2-3-4-5 \text{ with 6 attached to 3,}$$

$$E_7 : 1-2-3-4-5-6 \text{ with 7 attached to 3,}$$

$$E_8 : 1-2-3-4-5-6-7 \text{ with 8 attached to 3.}$$

The comarks are

$$E_6 : (1,2,3,2,1,2), \quad E_7 : (2,3,4,3,2,1,2), \quad E_8 : (2,4,6,5,4,3,2,3),$$
$$F_4 : (1,2,3,2), \qquad G_2 : (1,2).$$

The matrices $H_{\mathfrak{g}} = \ell_{\mathfrak{g}} W$ are

$$H_{E_6} = \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix},$$

$$H_{E_7} = \begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 2 & 4 \\ 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 3 \\ 4 & 8 & 12 & 9 & 6 & 3 & 7 \end{pmatrix},$$

$$H_{E_8} = \begin{pmatrix} 4 & 7 & 10 & 8 & 6 & 4 & 2 & 5 \\ 7 & 14 & 20 & 16 & 12 & 8 & 4 & 10 \\ 10 & 20 & 30 & 24 & 18 & 12 & 6 & 15 \\ 8 & 16 & 24 & 20 & 15 & 10 & 5 & 12 \\ 6 & 12 & 18 & 15 & 12 & 8 & 4 & 9 \\ 4 & 8 & 12 & 10 & 8 & 6 & 3 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 3 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 & 8 \end{pmatrix}.$$

For F_4 and G_2,

$$\begin{aligned} F_{F_4}(a_1,a_2,a_3,a_4) = {} & a_1^2 + 3a_2^2 + 6a_3^2 + 2a_4^2 \\ & + 3a_1a_2 + 4a_1a_3 + 2a_1a_4 + 8a_2a_3 \\ & + 4a_2a_4 + 6a_3a_4 + 11a_1 + 21a_2 + 30a_3 + 16a_4. \end{aligned}$$

$$F_{G_2}(s,t) = 2s^2 + 6st + 6t^2 + 10s + 18t.$$

The linear Newton coefficients are

$$u_{E_6} = (52, 100, 144, 100, 52, 72),$$
$$u_{E_7} = (72, 144, 216, 165, 112, 57, 105),$$
$$u_{E_8} = (96, 196, 300, 240, 180, 120, 60, 144),$$
$$u_{F_4} = (12, 24, 36, 18), \qquad u_{G_2} = (12, 24).$$

The data immediately give

$$\ell_{E_6} = 3, \quad \ell_{E_7} = 2, \quad \ell_{E_8} = 1, \quad \ell_{F_4} = 1, \quad \ell_{G_2} = 3.$$

For example,

$$Q_{2,E_6}(\omega_1) = \frac{52}{3}, \qquad Q_{2,E_7}(\omega_6) = \frac{57}{2}, \qquad Q_{2,G_2}(2\omega_1) = \frac{28}{3}.$$

B Classical coefficient formulas

For completeness, the following formulas provide the upper bounds in Proposition 3.4; the displayed fundamental values provide the matching lower bounds.

For A_r, with $h = r+1$,

$$Q_2(\omega_1) = \frac{r(r+2)}{r+1},$$

and with $F = hQ_2$,

$$u_i = (r+2)i(h-i), \qquad v_i = 2i(h-i), \qquad w_{ij} = 2i(h-j) \quad (i<j).$$

Thus $\ell_{A_r} = r+1$.

For B_r, let $d = \gcd(r,4)$ and $L = 4/d$. Then

$$Q_2(\omega_r) = \frac{r(2r+1)}{4},$$

and the coefficients of $F = LQ_2$ are

$$u_i = Li(2r-i+1), \quad v_i = 2Li, \quad w_{ij} = 2Li \quad (i<j<r),$$

$$w_{ir} = Li, \qquad u_r = \frac{r(2r+1)}{d}, \qquad v_r = \frac{2r}{d}.$$

Hence $\ell_{B_r} = 4/d$.

For C_r,

$$Q_2(\omega_1) = \frac{2r+1}{2},$$

and for $F = 2Q_2$,

$$u_i = i(2r-i+2), \qquad v_i = 2i, \qquad w_{ij} = 2i \quad (i<j).$$

Hence $\ell_{C_r} = 2$.

For D_r, let $d = \gcd(r,4)$ and $L = 4/d$. Then

$$Q_2(\omega_r) = \frac{r(2r-1)}{4},$$

and the coefficients of $F = LQ_2$ are

$$u_i = Li(2r-i), \quad v_i = 2Li, \quad w_{ij} = 2Li \quad (i<j\le r-2),$$

$$w_{i,r-1} = w_{ir} = Li, \quad u_{r-1} = u_r = \frac{r(2r-1)}{d},$$

$$v_{r-1} = v_r = \frac{2r}{d}, \qquad w_{r-1,r} = \frac{2(r-2)}{d}.$$

Thus $\ell_{D_r} = 4/d$.

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